PROJECTIVE SYSTEMS OF IMPRIMITIVITY

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Abstract

We apply Mackey procedure of classifying projective systems of imprimitivity to a thorough study of the projective unitary irreducible representations of the Galilei group in 1+3 and 1+2 dimensions.

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1 Introduction

Recently we have studied the projective unitary irreducible representations of the Poincaré group in 1+2 dimensions [1]. Here we provide the same analysis for the Galilei group in 1+2 dimensions.

The main difficulty in applying straighforwardly Mackey method is due to the rather complicated structure of the multiplier group (see [2] Appendix A, [3]). One can proceed in two ways. The first one mimicks the usual method of Mackey as presented, for instance in [4], and is based on the use of group extensions. This line of argument has been developped in [3]. The second method is based on the use of projective systems of imprimitivity and is also due to Mackey [5]. We will pursue the second method in this paper, but for the benefit of the reader, we will also outline the study of the projective systems of imprimitivity. We will prefer to follow the general line of argument and the notations of [4] and indicate the necessary changes. This will be done in Section 2. In Section 3 we apply the method to the Galilei group in 1+3 (only for illustrative purpose) and afterwards to the case of 1+2 dimensions.

2 Projective systems of imprimitivity

2.1 Let G be a Borel group acting in the Borel space X and let $\mathcal{H}$ be a Hilbert space. A *projective system of imprimitivity for G based on X and acting in $\mathcal{H}$* is a pair (U, P) where P ($E \mapsto P_E$) is a projector valued measure based on X and acting in $\mathcal{H}$, U ($g \mapsto U_g$) is a projective representation of G in $\mathcal{H}$ and such that the relations

$$U_g P_E U_g^{-1} = P_{g \cdot E} \tag{2.1}$$

are satisfied for all $g \in G$ and all Borel sets $E \subset X$. If U is a m-representation i.e. $m : G \times G \to \mathbf{T}$ (here $\mathbf{T}$ is the set of complex numbers of modulus 1) is the multiplier of U:

$$U_{g_1 g_2} = m(g_1, g_2)\, U_{g_1}\, U_{g_2} \tag{2.2}$$

for all $g_1, g_2 \in G$, then one also says that (U, P) is a *m-system of imprimitivity.* Now the notions of *equivalence, irreducibility, direct sum* and *commuting ring* for a m-system of imprimitivity are defined as for the usual case $m = 1$ (see [4], ch. VI, pp. 203).

2.2 If one wants to classify m-systems of imprimitivity the first step is to connect them with m-cocycles. Let G be a Borel group acting in the Borel space X and let $\mathcal{C}$ be an invariant measure class on X. Suppose M is a standard Borel group with the identity denoted by $\mathbf{1}$. (The usual case will be $M = \mathcal{U}(\mathcal{H}) \equiv$ the group of all the unitary operators in the Hilbert space $\mathcal{H}$). Finally, let $m : G \times G \to \mathbf{T}$ be a multiplier of G. We say that $f : G \times X \to M$ is a $m - (G, X, M)$-*cocycle* (or shortly a m-cocycle) if

(i) f is a Borel map
(ii) $f(e, x) = 1$ for almost all $x \in X$

(iii) $f(g_1 g_2, x) = m(g_1, g_2) \quad f(g_1, g_2 \cdot x) \quad f(g_2, x)$ for almost all $(g_1, g_2, x) \in G \times G \times X$.

The notions of *strict cocycle*, *(strict) cohomology*, *(strict) coboundary*, *(strict) cohomology class* are defined as for the usual case $m = 1$. We denote, as usual, (strict) equivalence by $(\approx) \sim$.

2.3 The connection between m-systems of imprimitivity and m-cocycles goes practically unchanged, as in [4]. So, lemma 6.6 from [4] trivially modifies in:

Lemma 1 *Let G be a Borel group verifying the second axiom of countability and $\mathcal{H}$ a separable Hilbert space. Suppose that L $(g \rightarrow L_g)$ is a mapping of G into the set of bounded operators in $\mathcal{H}$ such that (i) for all $f, f' \in \mathcal{H}$, $g \mapsto < f, L_g f' >$ is a Borel map; (ii) L_g is unitary for almost all g; (iii) $L_{g_1 g_2} = m(g_1, g_2) L_{g_1} \ L_{g_2}$ almost everywhere in $G \times G$.*

Then there exists exactly one m-representation U of G in $\mathcal{H}$ such that $L_g = U_g$ for almost all g.

Theorem 6.7 of [4] goes into:

Theorem 1 *Let the notations be as above. Let $\mathcal{K}$ be a Hilbert space, $M = \mathcal{U}(\mathcal{K})$, X a G-Borel space, $\mathcal{C}$ a G-invariant measure class on X and $\alpha \in \mathcal{C}$ a measure. For each $g \in G$, let r_g be a Borel function which is a version of the Radon-Nycodim derivative $d\alpha/d\alpha^{g^{-1}}$. Suppose that ϕ is a $m - (G, X, M)$-cocycle relative to the measure class of α. Then, there exists a unique m-system of imprimitivity (U, P) acting in $\mathcal{H} \equiv L^2(X, \mathcal{K}, \alpha)$ such that:*

$$P_E f = \chi_E \ f, \quad \forall f \in \mathcal{H} \tag{2.3}$$

and for almost all $g \in G$ we have

$$(U_g \ f)(x) = \left\{ r_g(g^{-1} \cdot x) \right\}^{1/2} \ \phi(g, g^{-1} \cdot x) \ f(g^{-1} \cdot x) \tag{2.4}$$

for all $f \in \mathcal{H}$ and for almost all $x \in X$. Moreover, the equivalence class of (U, P) depends only on the measure class $\mathcal{C}$ and on the cohomology class of ϕ.

2.4 Conversely, going from m-sytems of imprimitivity to m-cocycles involves again only a slight departure from [4]. So, lemma 6.10 of [4] stays true:

Lemma 2 *If the system of imprimitivity (U, P) is irreducible, then the measure class $\mathcal{C}$ is ergodic. If $\mathcal{C}$ is ergodic, then P is homogeneous.*

Next, suppose that $P = P(\{\mathcal{K}_n\}, \{\alpha_n\})$ is the decomposition of P into homogeneous projection valued measures. So $\mathcal{K}_n$ is a n-dimensional Hilbert space, $n \in \mathbb{N} \cup \{\infty\}$, $\alpha_1, \alpha_2, ..., \alpha_\infty$ are mutually singular σ-finite measures on X and $\alpha = \sum_{n=1}^{\infty} \alpha_n$. In the Hilbert space $\mathcal{H}_{n,\alpha} = L^2(X, \mathcal{K}_n, \alpha)$ we can construct a m-system of imprimitivity if we have at our disposal a $m - (G, X, M)$-cocycle; indeed we simply use Theorem 1 and define P and U according to (2.3) and (2.4) respectively. We denote in this case P by $P^{n,\alpha}$.

Then we have the counterpart of Theorem 6.11 of [4]:

Theorem 2 *Let (U, P) be a m-system of imprimitivity acting in the Hilbert space $\mathcal{H}$ such that P is homogoneous of multiplicity n $(1 \leq n \leq \infty)$. Let α be a σ-finite measure in the measure class $\mathcal{C}$ of P. Then (U, P) is equivalent to a m-system of imprimitivity $(U', P^{n,\alpha})$ acting in $\mathcal{H}_{n,\alpha}$. Moreover, there exists a one-one correspondence between the set of all cohomology classes of $m-(G, X, \mathcal{U}(\mathcal{K}_n))$-cocycles relative to $\mathcal{C}_\mathcal{P}$ and the set of all equivalence classes of m-systems of imprimitivity of the form $(U', P^{n,\alpha})$.*

2.5 From the preceeding two subsections it is clear that one is reduced to the classification problem for $m - (G, X, M)$-cocycles. As in [5], [4] this can be done rather completely in the case of a transitive action of G on X. In this case, one chooses $x_0 \in X$ arbitrary; it is known that the stability subgroup $G_0 \equiv G_{x_0}$ is closed and we have $X \simeq G/G_0$ as Borel spaces. So, from now on $X = G/G_0$ for some closed subgroup G_0. We denote by $\beta : G \to X$ the canonical projection: $\beta(g) = g \cdot G_0$; it is known that β is a Borel map. Next one notices that if α_0 is a σ-finite G-quasi-invariant measure on G, then one can produce a σ-finite G-quasi-invariant measure on X according to the formula:

$$\alpha(A) = \alpha_0(\beta^{-1}(A)) \tag{2.5}$$

for all Borel subsets $A \subset X$.

It is known that on a homogeneous G-space (as $X = G/G_0$) there exists a unique G-invariant measure class. So, if we take α_0 to be the Haar measure, α from (2.5) will give us a representative from this measure class. So, from now on, when speaking of cocycles on X we will always mean that they are relative to this measure class.

2.6 We concentrate for the moment on strict $m - (G, X, M)$-cocycles. Let X be a transitive G-space and f a strict $m - (G, X, M)$-cocycle. If $x \in X$, let us define $D : G_x \to M$ by:

$$D(h) \equiv f(h, x). \tag{2.6}$$

Then D is a m-representation of the group G_x in M. One calls D the *m-homomorphism defined by f in $x \in X$*. Then we have two elementary results which generalize lemma 5.23 of [4].

Lemma 3 *Let f be a (strict) $m - (G, X, M)$-cocycle. We define $f^0 : G \times G \to M$ by:*

$$f^0(g, g') \equiv f(g, \beta(g')), \quad \forall g, g' \in G. \tag{2.7}$$

Then f^0 is a (strict) $m - (G, G, M)$-cocycle. Conversely, let F be a (strict) $m - (G, G, M)$-cocycle, $G_0 \subset G$ a closed subgroup and $X = G/G_0$. Then there exists a (strict) $m - (G, X, M)$-cocycle f such that $f^0 = F$ iff:

$$F(g, g'h) = F(g, g'), \quad \forall g, g' \in G, \; \forall h \in G_0. \tag{2.8}$$

Moreover this correspondence $f \leftrightarrow F$ is one-one.

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Lemma 4 *If F ia a strict $m - (G, G, M)$-cocycle, there exists a unique Borel map $b : G \to M$ such that:*

$$F(g, g') = m(g, g')^{-1} \ b(g, g') \ b(g')^{-1}. \tag{2.9}$$

Next, we come to a succesion of results generalizing lemma 5.24 of [4].

Lemma 5 *A strict $m - (G, G, M)$-cocycle F verifies (2.8) for some closed subgroup $G_0 \subset G$ iff the map $b : G \to M$ associated as above verifies:*

$$b(e) = \mathbf{1} \tag{2.10}$$

$$b(gh) = m(g, h) \ b(g) \ b(h). \tag{2.11}$$

In this case the map $D : G \to M$ defined by

$$D(h) \equiv b(h) \tag{2.12}$$

is a m-representation of G_0 in M.

Lemma 6 *In the conditions above let f be a strict $m - (G, X, M)$-cocycle, f^0 the associated strict $m - (G, G, M)$-cocycle and $D : G_0 \to M$ the map corresponding to f^0 as above. Then f defines D in $G_0 \in G/G_0$.*

Lemma 7 *There is a one-one correspondence between strict $m - (G, X, M)$-cocycles and maps $b : G \to M$ verifying (2.10) and (2.11) above.*

One can construct maps verifying the conditions above using Borel cross sections i. e. Borel maps $c : X \to G$ verifying

$$\beta \circ c = id. \tag{2.13}$$

Lemma 8 *Let $G_0 \subset G$ be a closed subgroup and $D : G_0 \to M$ a m-representation. Then there exists a Borel map $b : G \to M$ such that $b|_{G_0} = D$ and*

$$b(gh) = m(g, h) \ b(g) \ b(h), \quad \forall g \in G, \ \forall h \in G_0. \tag{2.14}$$

Proof

Indeed, one knows that cross sections $c : X \to G$ do exist. One first arranges such that:

$$c(G_0) = e. \tag{2.15}$$

Then, one defines $a : G \to G$ according to:

$$a(g) \equiv c(\beta(g))^{-1} g. \tag{2.16}$$

In fact $a : G \to G_0$. If $D : G_0 \to M$ is a m-representation one finally checks that the map

$$b(g) \equiv m(c(\beta(g))^{-1}, g)^{-1} \ D(a(g)) \tag{2.17}$$

verifies the conditions in the statement of the lemma. $\square$

Lemma 9 *Let f_1, f_2 be two strict $m - (G, X, M)$-cocycles and D_1, D_2 the associated m-representations. Then f_1 and f_2 are cohomologous ($f_1 \approx f_2$) iff D_1 and D_2 are equivalent m-representations.*

Lemma 10 *Let $D : G_0 \to M$ be a m-representation and $b_1, b_2 : G \to M$ two Borel maps verifying $b_i|_{G_0} = D$ and $b_i(gh) = b_i(g) \, b_i(h), \quad \forall g \in G, \forall h \in G_0 \ (i = 1, 2)$. Let f_1, f_2 be the strict $m - (G, X, M)$-cocycles associated to b_1, b_2 respectively (see lemma 7). Then $f_1 \approx f_2$.*

In conclusion we have:

Proposition 1 *There exists a one-one correspondence between the set of cohomology classes of strict $m - (G, X, M)$-cocycles and the set of equivalence classes of $m|_{G_0 \times G_0}$-representations of G_0 in M. More precisely, let $D : G_0 \to M$ be a $m|_{G_0 \times G_0}$-representation. Then every strict $m - (G, X, M)$-cocycle which defines D in G_0 is strictly cohomologous to:*

$$\phi(g, x) = m(c(g \cdot x)^{-1}, g)^{-1} \, m(c(g \cdot x)^{-1} gc(x), c(x)^{-1}) D(c(g \cdot x)^{-1} gc(x)). \quad (2.18)$$

Needless to say, all the computations involved in proving the results above are elementary.

2.7 We analyse here the general case of $m - (G, X, M)$-cocycles. The key point is to generalize lemma 5.26 of [4]. We do this in some detail.

Lemma 11 *If f is any $m - (G, X, M)$-cocycle there exists a strict $m - (G, X, M)$-cocycle f_1 such that*

$$f(g, x) = f_1(g, x) \qquad (2.19)$$

for almost all $(g, x) \in G \times X$; f_1 is uniquely determined up to strict cohomology.

Proof Follows [4].

If f is a $m - (G, X, M)$-cocycle, let f^0 be the associated $m - (G, G, M)$-cocycle (see lemma 3). We have

$$f^0(g_1 g_2, g_3) = m(g_1, g_2) \, f^0(g_1, g_2 g_3) \, f^0(g_2, g_3)$$

for almost all $(g_1, g_2, g_3) \in G \times G \times G$. Hence there exists a $g_0 \in G_0$ such that

$$f^0(g_1 g_2, g_0) = m(g_1, g_2) \, f^0(g_1, g_2 g_0) \, f^0(g_2, g_0)$$

for almost all $(g_1, g_2) \in G \times G$. We take $g = g_1, \ g' = g_2 g_0$ and conclude that

$$f^0(g, g') = m(g, g' g_0^{-1})^{-1} \, f^0(gg' g_0^{-1}, g_0) \, f^0(g' g_0^{-1}, g_0)^{-1}$$

or, using the multiplier identity:

$$f^0(g, g') = m(g, g')^{-1} \, m(g', g_0^{-1}) \, m(gg', g_0^{-1}) \, f^0(gg' g_0^{-1}, g_0) \, f^0(g' g_0^{-1}, g_0)^{-1}$$

for almost all $(g, g') \in G \times G$.

We define the map $d_0 : G \to M$ by

$$d_0(g) \equiv m(g, g')^{-1} \ f^0(gg_0^{-1}, g_0).$$

Then d_0 is a Borel map and we have:

$$f^0(g, g') = m(g, g')^{-1} \ d_0(gg') \ d_0(g')^{-1}$$

for almost all $(g, g') \in G \times G$. If we redefine d_0 :

$$d(g) \equiv d_0(g) \ d_0(e)^{-1}$$

then we have:

$$d(e) = \mathbf{1}$$

and

$$f^0(g, g') = m(g, g')^{-1} \ d(gg') \ d(g')^{-1} \tag{2.20}$$

for almost all $(g, g') \in G \times G$.

Now let $h \in G_0$ be arbitrary. We have then:

$$f^0(g, g'h) = f^0(g, g').$$

Inserting here (2.20) we obtain:

$$m(gg', h)^{-1} \ d(gg')^{-1} \ d(gg'h) = m(g', h)^{-1} \ d(g')^{-1} \ d(g'h)$$

for almost all $(g, g') \in G \times G$. Because the measure class $\mathcal{C}$ on $X = G$ is known to be ergodic, it follows that the function: $g' \mapsto m(g', h)^{-1} \ d(g')^{-1} \ d(g'h)$ is almost everywere constant i.e. we have $D : G_0 \to M$ such that;

$$d(gh) = m(g, h) \ d(g) \ D(h) \tag{2.21}$$

for almost all $g \in G$; this fixes D uniquely. One easily finds out that;

$$D(e) = \mathbf{1}$$

$$D(h_1 h_2) = m(h_1, h_2) \ D(h_1) \ D(h_2)$$

i.e D is a $m|_{G_0 \times G_0}$-representation of G_0 in M.

We prove now that D is a Borel map. Indeed, let λ be a quasi-invariant measure on G normalized by $\lambda(G) = 1$. Then from (2.21) one has:

$$D(h) = \int d\lambda(g) \ m(g, h)^{-1} \ d(g)^{-1} \ d(gh).$$

Applying lemma 8 one can find out a Borel map $d' : G \to M$ such that:

$$d'(gh) = m(g, h) \ d'(g) \ D(h).$$

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Comparing with (2.21) it follows that $\forall h \in G_0$:

$$d(g) \ d'(g)^{-1} = d(gh) \ d'(gh)$$

for almost all $g \in G$. Like in [4] it follows that there exists a Borel map $k : X \to M$ such that

$$d(g) = k(\beta(g)) \ d'(g)$$

for almost all $g \in G$. If we define $d_1 : G \to M$ by:

$$d_1 \equiv k(\beta(g)) \ d'(g)$$

then we obviously have:

$$d(g) = d_1(g)$$

for almost all $g \in G$ and also:

$$d_1(gh) = m(g,h) \ d_1(g) \ D(h), \quad \forall g \in G, \ \forall h \in G_0$$

so, according to lemma 7 there exists a strict $m - (G, X, M)$-cocycle f_1 such that

$$f_1^0(g, g') = m(g, g')^{-1} \ d_1(gg') \ d_1(g')^{-1}.$$

It is clear that $f = f_1$ almost everywere in $G \times X$. □
As corollaries we have as in [4]

Lemma 12 *Let f_i $(i = 1, 2)$ be two strict $m - (G, X, M)$-cocycles. Then $f_1 \sim f_2 \Leftrightarrow f_1 \approx f_2$.*

Lemma 13 *Let γ be a cohomology class of $m - (G, X, M)$-cocycles and $\gamma_{st} \subset \gamma$ the set of strict $m - (G, X, M)$-cocycles. Then γ_{st} is non-void and is a strict class of cohomology.*

Lemma 14 *There is a one-one correspondence between the set of equivalence classes of transitive m-systems of imprimitivity for G in $\mathcal{H}_{n,\alpha}$ and the set of strict cohomology classes of $m - (G, X, \mathcal{U}(\mathcal{K}_n))$-cocycles relative to $\mathcal{C}_{\mathcal{P}}$.*

2.8 In conclusion, we formulate the main results.

Proposition 2 *There is a one-one correspondence between the set of equivalence classes of transitive m-systems of imprimitivity for G based on X and the set of equivalence classes of $m|_{G_0 \times G_0}$-representations of G_0 in M.*

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Proposition 3 *Let $D : G_0 \to M$ be a m-representation and (U, P) a m-system of imprimitivity associated to D in the sense of the preceeding proposition. Then the commuting ring of D is isomorphic to the commuting ring of (U, P). Hence, the correspondence from the preceeding proposition preserves irreducibility and the direct sum.*

The proofs of these propositions are similar to those in [4]. One has to notice that all the multiplier factors in various formulæ conveniently cancel.

So, the procedure of constructing irreducible m-systems of imprimitivity consists in following three steps.

(A) First, one notes that from lemma 2 it easily follows that the m-system of imprimitivity is transitive. So, one classifies all G-orbits in X. Let us fix now a certain orbit $\mathcal{O}$. We have to classify the irreducible m-systems of imprimitivity for G based on $X = \mathcal{O}$. One identifies some closed subgroup G_0 such that $X \simeq G/G_0$ and lists the irreducible $m|_{G_0 \times G_0}$-representations up to unitary equivalence.

(B) One computes an associated $m - (G, X, M)$-cocycle using, eventually, the formula (2.18).

(C) One constructs the m-system of imprimitivity according to (2.3) and (2.4) (see theorem 1).

3 The Projective Unitary Irreducible Representations of the Galilei Group

3.1 By definition the ortochronous Galilei group in $1 + n$ dimensions $\mathcal{G}^\uparrow$ is set-theoretically $O(n-1) \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1}$ with the composition law:

$$(R_1, \mathbf{v}_1, \eta_1, \mathbf{a}_1) \cdot (R_2, \mathbf{v}_2, \eta_2, \mathbf{a}_2) = (R_1 R_2, \mathbf{v}_1 + R_1 \mathbf{v}_2, \eta_1 + \eta_2, \mathbf{a}_1 + R_1 \mathbf{a}_2 + \eta_2 \mathbf{v}_1). \quad (3.1)$$

We organize $\mathbb{R}^{n-1}$ as column vectors, $O(n-1)$ as $(n-1) \times (n-1)$ real orthogonal matrices and we use consistently matrix notations. This group acts naturally on $\mathbb{R} \times \mathbb{R}^{n-1}$ as follows

$$(R, \mathbf{v}, \eta, \mathbf{a}) \cdot (T, \mathbf{X}) = (T + \eta, R\mathbf{X} + T\mathbf{v} + \mathbf{a}). \quad (3.2)$$

The proper orthochronous Galilei group $\mathcal{G}_+^\uparrow$ is by definition:

$$\mathcal{G}_+^\uparrow \equiv \{(R, \mathbf{v}, \eta, \mathbf{a}) | det(R) = 1\}. \quad (3.3)$$

The groups $\mathcal{G}^\uparrow$ and $\mathcal{G}_+^\uparrow$ are Lie groups.

As it is well known, the classification of all projective unitary irreducible representations of $\mathcal{G}_+^\uparrow$ is done following the steps below:

1) One identifies the universal covering group $\widetilde{\mathcal{G}_+^\uparrow}$ of $\mathcal{G}_+^\uparrow$. Let us denote by $\pi : \widetilde{\mathcal{G}_+^\uparrow} \to \mathcal{G}_+^\uparrow$ the canonical projection. If V is a projective (unitary irreducible) representation of $\mathcal{G}_+^\uparrow$ then

$$\tilde{V} \equiv V \circ \pi \quad (3.4)$$

is a projective (unitary irreducible) representation of $\widetilde{\mathcal{G}_+^\uparrow}$ verifying:

$$\tilde{V}(g_0) = \lambda \times id \tag{3.5}$$

for any $g_0 \in Ker(\pi)$. Here λ is a complex number of modulus 1.

It will be clear immediately why it is more convenient to classify, up to unitary equivalence, the projective unitary irreducible representations of $\widetilde{\mathcal{G}_+^\uparrow}$ verifying the condition (3.5).

2) One determines the cohomology group $H^2(\widetilde{\mathcal{G}_+^\uparrow}, \mathbb{R})$ using infinitesimal arguments. This is possible because for connected and simply connected Lie groups one has $H^2(G, \mathbb{R}) \simeq \mathrm{H}^2(\mathrm{Lie(G)}, \mathbb{R})$.

3) One selects a multiplier from every cohomology class and tries to classify the unitary irreducible m-representations of $\widetilde{\mathcal{G}_+^\uparrow}$ and afterwards selects those verifying the condition (3.5). This can be done as in [3] generalizing theorem 7.16 of [4]. Alternatively, one could try to connect the m-representations to some projective systems of imprimitivity. This approach will be used in the following.

We will start with the Galilei group in 1+3 dimensions and afterwards we will examine in detail the more interesting case of 1+2 dimensions.

3.2 Let $\mathcal{G}_+^\uparrow$ the proper orthochronous Galilei group in 1+3 dimensions.

1) The universal covering group is [4] set-theoretically: $SU(2) \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3$ with the composition law:

$$(A_1, \mathbf{v}_1, \eta_1, \mathbf{a}_1) \cdot (A_2, \mathbf{v}_2, \eta_2, \mathbf{a}_2) = (A_1 A_2, \mathbf{v}_1 + \delta(A_1)\mathbf{v}_2, \eta_1 + \eta_2, \mathbf{a}_1 + \delta(A_1)\mathbf{a}_2 + \eta_2 \mathbf{v}_1). \tag{3.6}$$

Here $\delta : SU(2) \to SO(3)$ is the covering map described by:

$$[\delta(A) \cdot x] = A[x]A^* \tag{3.7}$$

where

$$[x] \equiv x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3 \tag{3.8}$$

($\sigma_1, \sigma_2, \sigma_3$ being the Pauli matrices). The covering map δ extends trivially to a map (denoted also by δ) from $\widetilde{\mathcal{G}_+^\uparrow}$ onto $\mathcal{G}_+^\uparrow$ which is the looked for covering map.

2) In [4] theorem 7.42 describes the multiplier group of $\widetilde{\mathcal{G}_+^\uparrow}$. Namely, every such multiplier is cohomologous to one of the following form:

$$m_\tau(r_1, r_2) \equiv exp\left\{i\frac{\tau}{2}\left[\mathbf{a}_1 \cdot \delta(A_1)\mathbf{v}_2 - \mathbf{v}_1 \cdot \delta(A_1)\mathbf{a}_2 + \eta_2 \mathbf{v}_1 \cdot \delta(A_1)\mathbf{v}_2\right]\right\}. \tag{3.9}$$

Here $r_i = (A_i, \mathbf{v}_i, \eta_i, \mathbf{a}_i)$, $(i = 1, 2)$ and $\tau \in \mathbb{R}$. Moreover $m_{\tau_1} \sim m_{\tau_2}$ iff $\tau_1 = \tau_2$.

3) Let V be a (unitary) m_τ-representation of $\mathcal{G}_+^\uparrow$. We define:

$$U_{A,\mathbf{v}} \equiv V_{A,\mathbf{v},0,\mathbf{0}} \tag{3.10}$$

$$W_{\eta,\mathbf{a}} \equiv V_{\mathbf{I},\mathbf{0},\eta,\mathbf{a}}. \tag{3.11}$$

Then we have

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Proposition 4 *(i) U and W are (unitary) representations of $G = SU(2) \times_\delta \mathbb{R}^3$ and respectively $\mathbb{R} \times \mathbb{R}^3$ i.e.*

$$U_{A_1,\mathbf{v}_1} \ U_{A_2,\mathbf{v}_2} = U_{A_1 A_2,\mathbf{v}_1+\delta(A_1)\mathbf{v}_2} \tag{3.12}$$

$$W_{\eta_1,\mathbf{a}_1} \ W_{\eta_2,\mathbf{a}_2} = W_{\eta_1+\eta_2,\mathbf{a}_1+\mathbf{a}_2}. \tag{3.13}$$

One also has:

$$U_{A,\mathbf{v}} \ W_{\eta,\mathbf{a}} \ U_{A,\mathbf{v}}^{-1} = exp\left\{ i\tau \left[\mathbf{v}\cdot\delta(A)\mathbf{a} + \eta\frac{\mathbf{v}^2}{2} \right] \right\} W_{\eta,\delta(A)\mathbf{a}+\eta\mathbf{v}} \tag{3.14}$$

(ii) Conversely, if U and W are as above, let us define

$$V_{A,\mathbf{v},\eta,\mathbf{a}} \equiv exp\left\{ i\frac{\tau}{2}\mathbf{a}\cdot\mathbf{v} \right\} W_{\eta,\mathbf{a}} \ U_{A,\mathbf{v}}. \tag{3.15}$$

Then V is a (unitary) m_τ-representation of $\widetilde{\mathcal{G}_+^\uparrow}$.
(iii) Moreover, the commuting ring of V is isomorphic to the commuting ring of the pair (U,W).

We push the analysis a step further by considering the projector valued measure P associated to W:

$$W_{\eta,\mathbf{a}} = \int e^{i(p_0\eta+\mathbf{p}\cdot\mathbf{a})}dP(p_0,\mathbf{p}). \tag{3.16}$$

Then one easily proves that (3.14) is equivalent to:

$$U_{A,\mathbf{v}} \ P_E \ U_{A,\mathbf{v}}^{-1} = P_{(A,\mathbf{v})\cdot E} \tag{3.17}$$

where the action of G on $\mathbb{R} \times \mathbb{R}^2$ is:

$$(A,\mathbf{v})\cdot[p_0,\mathbf{p}] = \left[p_0 + \frac{\tau}{2}\mathbf{v}^2 - \mathbf{v}\cdot\delta(A)\mathbf{p}, \delta(A)\mathbf{p} - \tau\mathbf{v} \right]. \tag{3.18}$$

So (U,P) is a (true) system of imprimitivity. Moreover the commuting ring of V is isomorphic to the commuting ring of (U,P). To classify the unitary irreducible m_τ-representations of $\widetilde{\mathcal{G}_+^\uparrow}$, one classifies the irreducible systems of imprimitivity for G based on $\mathbb{R} \times \mathbb{R}^3$ relative to the action (3.18). According to the lemma 2 they correspond to the G-orbits relative to this action. Then one reconstructs W from (3.16) and uses (3.15) to obtain the m_τ-representation V. The classification of the systems of imprimitivity of G relative to (3.18) is easy because we do not have the complications due to the existence of the multipliers discussed in detail in the preceeding chapter. We will consider only the physically interesting case $\tau \neq 0$.

The orbits of (3.18) are:

$$Z_\rho = \{[p_0,\mathbf{p}]|\mathbf{p}^2 - 2\tau p_0 = \rho\} \quad (\rho \in \mathbb{R}).$$

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So we study the systems of imprimitivity for $\widetilde{\mathcal{G}}_+^\uparrow$ on Z_ρ. The stability subgroup of $\left[-\frac{\rho}{2\tau}, \mathbf{0}\right]$ is:

$$G_0 = \{(A, \mathbf{0})|A \in SU(2)\} \simeq SU(2).$$

The unitary irreducible representations of G_0 are, up to unitary equivalence, the well known $D^{(s)}$ (with $s \in \mathbb{N}/2$). A convenient associated cocycle is simply:

$$\phi^{(s)}((A, \mathbf{v}), [p_0, \mathbf{p}]) = D^{(s)}(A). \tag{3.19}$$

If we identify $Z_\rho \simeq \mathbb{R}^3$ via $\left[\frac{1}{2\tau}(\mathbf{p}^2 - \rho), \mathbf{p}\right] \leftrightarrow \mathbf{p}$ and consider on $\mathbb{R}^3$ the Lebesgue $\widetilde{\mathcal{G}}_+^\uparrow$-invariant measure $d\mathbf{p}$, then it follows that the corresponding system of imprimitivity acts in $\mathcal{H} = L^2(\mathbb{R}^3, \mathbf{C}^{2s+1}, d\mathbf{p})$ according to:

$$
\begin{align}
(U_{A,\mathbf{v}}\, f)\,(\mathbf{p}) &= D^{(s)}(A)\, f(\delta(A)^{-1}(\mathbf{p} + \tau\mathbf{v})) \tag{3.20} \\
P_E &= \chi_E. \tag{3.21}
\end{align}
$$

One reconstructs W immediately:

$$(W_{\eta,\mathbf{a}}\, f)\,(\mathbf{p}) = exp\left\{i\left(\frac{\mathbf{p}^2 - \rho}{2\tau}\eta + \mathbf{p}\cdot\mathbf{a}\right)\right\}\, f(\mathbf{p}) \tag{3.22}$$

and, using (3.15) gets the expression of the m_τ-representation as:

$$
\begin{align}
(V_{A,\mathbf{v},\eta,\mathbf{a}}\, f)\,(\mathbf{p}) &= exp\left\{i\left(\frac{\tau}{2}\mathbf{v}\cdot\mathbf{a} + \frac{\mathbf{p}^2}{2\tau}\eta - \frac{\rho}{2\tau}\eta + \mathbf{p}\cdot\mathbf{a}\right)\right\} \\
&\quad D^{(s)}(A)\, f(\delta(A)^{-1}(\mathbf{p} + \tau\mathbf{v})) \tag{3.23}
\end{align}
$$

For obvious reasons the factor $\left\{-i\frac{\rho}{2\tau}\eta\right\}$ can be discarded and we reobtain the results of ch. IX,8 of [4].

Finally, one notices that the condition (3.5) is fulfilled. So, the unitary irreducible m_τ-representations of $\widetilde{\mathcal{G}}_+^\uparrow$ are indexed, beside $\tau \in R^*$ by $s \in \mathbb{N}/2$, are given by (3.23) (with $\rho = 0$) and induce projective unitary irreducible representations of $\mathcal{G}_+^\uparrow$.

3.3 Let $\mathcal{G}_+^\uparrow$ be the proper orthochronous Galilei group in 1+2 dimensions.

1) The covering group $\widetilde{\mathcal{G}}_+^\uparrow$ is the set $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ with the composition law:

$$
\begin{align}
(x_1, \mathbf{v}_1, \eta_1, \mathbf{a}_1) \cdot (x_2, \mathbf{v}_2, \eta_2, \mathbf{a}_2) &= \\
(x_1 + x_2, \mathbf{v}_1 + \delta(x_1)\mathbf{v}_2, \eta_1 + \eta_2, \mathbf{a}_1 + \delta(x_1)\mathbf{a}_2 + \eta_2\mathbf{v}_1), \tag{3.24}
\end{align}
$$

where $\delta : \mathbb{R} \to SO(2)$ is the covering map described by:

$$\delta(x) \equiv \begin{pmatrix} cos(x) & sin(x) \\ -sin(x) & cos(x) \end{pmatrix}. \tag{3.25}$$

The map δ extends obviously to the covering map (denoted also by δ) from $\widetilde{\mathcal{G}}_+^\uparrow$ onto $\mathcal{G}_+^\uparrow$.

2) The multiplier group for $\widetilde{\mathcal{G}_+^\uparrow}$ is described in [3]. One first identifies $Lie(\widetilde{\mathcal{G}_+^\uparrow}) \simeq \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ with the Lie bracket:

$$[(\alpha_1, \mathbf{u}_1, t_1, \mathbf{x}_1), (\alpha_2, \mathbf{u}_2, t_2, \mathbf{x}_2)] =$$

$$(0, A(\alpha_1 \mathbf{u}_2 - \alpha_2 \mathbf{u}_1), 0, A(\alpha_1 \mathbf{x}_2 - \alpha_2 \mathbf{x}_1) + t_2 \mathbf{u}_1 - t_1 \mathbf{u}_2). \qquad (3.26)$$

Here

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Next, one computes by simple algebraic manipulations $H^2(Lie(\widetilde{\mathcal{G}_+^\uparrow}), \mathbb{R})$.

One finds out that every Lie algebraic cocycle ξ is cohomologous to one of the form $\tau \xi_0 + F \xi_1 + S \xi_2$ where

$$\xi_0((\alpha_1, \mathbf{u}_1, t_1, \mathbf{x}_1), (\alpha_2, \mathbf{u}_2, t_2, \mathbf{x}_2)) = \mathbf{x}_1 \cdot \mathbf{u}_2 - \mathbf{x}_1 \cdot \mathbf{u}_2 \qquad (3.27)$$

$$\xi_0((\alpha_1, \mathbf{u}_1, t_1, \mathbf{x}_1), (\alpha_2, \mathbf{u}_2, t_2, \mathbf{x}_2)) = \frac{1}{2} < \mathbf{u}_1, \mathbf{u}_2 > \qquad (3.28)$$

$$\xi_0((\alpha_1, \mathbf{u}_1, t_1, \mathbf{x}_1), (\alpha_2, \mathbf{u}_2, t_2, \mathbf{x}_2)) = \alpha_1 t_2 - \alpha_2 t_1. \qquad (3.29)$$

Here $< \cdot, \cdot >$ is the sesquilinear form on $\mathbb{R}^2$ given by:

$$< \mathbf{u}, \mathbf{v} > \equiv \mathbf{u}^t A \mathbf{v}. \qquad (3.30)$$

Moreover $\tau_1 \xi_0 + F_1 \xi_1 + S_1 \xi_2 \sim \tau_2 \xi_0 + F_2 \xi_1 + S_2 \xi_2$ iff $\tau_1 = \tau_2$, $F_1 = F_2$, $S_1 = S_2$.

Next, one determines $H^2(\widetilde{\mathcal{G}_+^\uparrow}, \mathbb{R})$ and finds out that every multiplier of $\widetilde{\mathcal{G}_+^\uparrow}$ is cohomologous to one of the following form:

$$m_{\tau,F,S}(r_1, r_2) \equiv exp\left\{ i\frac{\tau}{2}[\mathbf{a}_1 \cdot \delta(x_1)\mathbf{v}_2 - \mathbf{v}_1 \cdot \delta(x_1)\mathbf{a}_2 + \eta_2 \mathbf{v}_1 \cdot \delta(x_1)\mathbf{v}_2] \right\}$$

$$\times \quad exp\left\{ \frac{iF}{2} < \mathbf{v}_1, \delta(x_1)\mathbf{v}_2 > + iS\eta_1 x_2 \right\} \qquad (3.31)$$

Here $r_i = (x_i, \mathbf{v}_i, \eta_i, \mathbf{a}_i)$ ($i = 1, 2$). Moreover $m_{\tau_1, F_1, S_1} \sim m_{\tau_2, F_2, S_2}$ iff $\tau_1 = \tau_2$, $F_1 = F_2$, $S_1 = S_2$.

3) Let V be a (unitary) $m_{\tau,F,S}$-representation of $\widetilde{\mathcal{G}_+^\uparrow}$. We define

$$U_{x,\mathbf{v}} \equiv V_{x,\mathbf{v},0,\mathbf{0}} \qquad (3.32)$$

$$W_{\eta,\mathbf{a}} \equiv V_{0,\mathbf{0},\eta,\mathbf{a}}. \qquad (3.33)$$

Then we have

Proposition 5 *(i)* U *is a (unitary)* $m_F \equiv m_{0,F,0}$*-representation of* $G \equiv \mathbb{R} \times_\delta \mathbb{R}^2$:

$$U_{x_1,\mathbf{v_1}} \ U_{x_2,\mathbf{v_2}} = exp \left\{ \frac{iF}{2} < \mathbf{v_1}, \delta(x_1)\mathbf{v_2} > \right\} U_{x_1+x_2,\mathbf{v_1}+\delta(x_1)\mathbf{v_2}}. \tag{3.34}$$

W is a (unitary) representation of the Abelian group $\mathbb{R} \times \mathbb{R}^2$ *i.e.*

$$W_{\eta_1,\mathbf{a_1}} \ W_{\eta_2,\mathbf{a_2}} = W_{\eta_1+\eta_2,\mathbf{a_1}+\mathbf{a_2}}. \tag{3.35}$$

One also has:

$$U_{x,\mathbf{v}} \ W_{\eta,\mathbf{a}} \ U_{x,\mathbf{v}}^{-1} = exp \left\{ i\tau \left[\mathbf{v} \cdot \delta(x)\mathbf{a} + \eta \frac{\mathbf{v}^2}{2} \right] + iS\eta x \right\} W_{\eta,\delta(x)\mathbf{a}+\eta\mathbf{v}} \tag{3.36}$$

(ii) Conversely, if U and W are as above, let us define:

$$V_{x,\mathbf{v},\eta,\mathbf{a}} \equiv exp \left\{ i\frac{\tau}{2}\mathbf{a} \cdot \mathbf{v} + iS\eta x \right\} W_{\eta,\mathbf{a}} \ U_{x,\mathbf{v}}. \tag{3.37}$$

Then V is a (unitary) $m_{\tau,F,S}$-representation of $\widetilde{\mathcal{G}_+^\uparrow}$.
(iii) Moreover, the commuting ring of V is isomorphic to the commuting ring of the couple (U, W).

Like in subsection 3.2, we consider the projector valued measure P associated to W:

$$W_{\eta,\mathbf{a}} = \int e^{i(\eta p_0 + \mathbf{a} \cdot \mathbf{p})} dP(p_0, \mathbf{p}). \tag{3.38}$$

Then one easily proves that (3.36) is equivalent to:

$$U_{x,\mathbf{v}} \ P_E \ U_{x,\mathbf{v}}^{-1} = P_{(x,\mathbf{v}) \cdot E} \tag{3.39}$$

where the action of G on $\mathbb{R} \times \mathbb{R}^2$ is

$$(x, \mathbf{v}) \cdot [p_0, \mathbf{p}] = \left[p_0 - \mathbf{v} \cdot \delta(x)\mathbf{p} + \frac{\tau\mathbf{v}^2}{2} - Sx, \delta(x)\mathbf{p} - \tau\mathbf{v} \right]. \tag{3.40}$$

So (U, P) is a m_F-system of imprimitivity. Moreover, the commuting ring of V is isomorphic to the commuting ring of (U, P). The classification of the unitary irreducible $m_{\tau,F,S}$-representations of $\widetilde{\mathcal{G}_+^\uparrow}$ is reduced to the classification of the irreducible m_F-systems of imprimitivity for G based on $\mathbb{R} \times \mathbb{R}^2$ and relative to the action (3.40). Then W is obtained from (3.38) and V from (3.37).

The orbits of (3.40) are easy to compute. We distinguish four cases which must be studied separatedly:
    a) $\tau \neq 0, \ S = 0$

$$Z_\rho^1 \equiv \{[p_0, \mathbf{p}] | \mathbf{p}^2 - 2\tau p_0 = \rho\}; \ \rho \in \mathbb{R},$$

b) $\tau \neq 0, \ S \neq 0$

$$Z^2 \equiv \{[p_0, \mathbf{p}] | \mathbf{p} \in \mathbb{R}^2, p_0 \in \mathbb{R}\}$$

c) $\tau = 0, \ S = 0$

$$Z_r^3 \equiv \{[p_0, \mathbf{p}] | p_0 \in \mathbb{R}, \mathbf{p}^2 = r^2\}; \ r \in \mathbb{R}_+$$
$$Z_{p_0}^4 \equiv \{[p_0, \mathbf{0}]\}; \ p_0 \in \mathbb{R}$$

d) $\tau = 0, \ S \neq 0$

$$Z_r^5 \equiv \{[p_0, \mathbf{p}] | \mathbf{p}^2 = r^2, p_0 \in \mathbb{R}\}; r \in \mathbb{R}_+ \cup \{0\}.$$

We analyse them one by one:

a) We have:

$$G_0 = G_{[-\frac{\rho}{2\tau}, \mathbf{0}]} = \{(x, \mathbf{0}) | x \in \mathbb{R}\} \simeq \mathbb{R}$$

Because $m|_{G_0 \times G_0} = 0$ we have to consider only the unitary irreducible representations of $\mathbb{R}$. They are $D^{(s)}$ $(s \in \mathbb{R})$ acting in $\mathbf{C}$ as follows:

$$D^{(s)}(x, \mathbf{0}) = e^{isx}. \tag{3.41}$$

As an associated cocycle we can take:

$$\phi^{(s)}((x, \mathbf{v}), [p_0, \mathbf{p}]) = e^{isx}. \tag{3.42}$$

The corresponding system of imprimitivity acts in $\mathcal{H} = L^2(\mathbb{R}^2, d\mathbf{p})$ if one notices that it is possible to identify $Z_\rho^1 \simeq \mathbb{R}^2$ with the Lebesgue measure $d\mathbf{p}$, like in subsection 3.2 One has:

$$(U_{x, \mathbf{v}} \ f)(\mathbf{p}) = e^{isx} f(\delta(x)^{-1}(\mathbf{p} + \tau \mathbf{v})) \tag{3.43}$$

$$P_E = \chi_E. \tag{3.44}$$

We reconstruct W as in subsection 3.2 (see (3.22)) and then the representation V. One gets in this way the expression:

$$\left(V_{(x, \mathbf{v}, \eta, \mathbf{a})} \ f\right)(\mathbf{p}) = exp\left\{i\left(\frac{\tau}{2}\mathbf{a} \cdot \mathbf{v} + \frac{\mathbf{p}^2 - \rho}{2\tau} + \mathbf{a} \cdot \mathbf{p}\right)\right\}$$
$$\times \ exp\left\{i\left(\frac{F}{2\tau} <\mathbf{v}, \mathbf{p}> +sx\right)\right\} f(\delta(x)^{-1}(\mathbf{p} + \tau \mathbf{v}) \tag{3.45}$$

Like in 3.2 one can discard the factor $exp\left(-i\eta\frac{\rho}{2\tau}\right)$. The condition (3.5) is fulfilled so we have obtained a family of $m_{\tau, F, 0}$-representations depending beside τ, F on $s \in \mathbb{R}$.

b) In this case we have:

$$G_0 = G_{[0, \mathbf{0}]} = \{(0, \mathbf{0})\}$$

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and it is clear that one has to consider only the trivial representation of G_0. The corresponding m_F-system of imprimitivity acts in $L^2(\mathbb{R} \times \mathbb{R}^2, \mathrm{d}p_0 \otimes \mathrm{d}\mathbf{p})$ according to:

$$(U_{x,\mathbf{v}} \, f) \, (p_0, \mathbf{p}) = f(p_0 + \mathbf{v} \cdot \mathbf{p} + \frac{1}{2}\tau\mathbf{v}^2 + Sx, \delta(x)^{-1}(\mathbf{p} + \tau\mathbf{v})) \tag{3.46}$$

$$P_E = \chi_E. \tag{3.47}$$

The reconstruction of W is immediate:

$$(W_{\eta,\mathbf{a}} \, f) \, (p_0, \mathbf{p}) = e^{i(\eta p_0 + \mathbf{a} \cdot \mathbf{p})} \, f(p_0, \mathbf{p}) \tag{3.48}$$

and using (3.37), the expression of a $m_{\tau,F,S}$-representation of $\widetilde{\mathcal{G}_+^\uparrow}$ is obtained:

$$
\begin{aligned}
(V_{x,\mathbf{v},\eta,\mathbf{a}}f) \, (p_0, \mathbf{p}) \;=\; & exp\left\{i\left[\frac{\tau}{2}\mathbf{a} \cdot \mathbf{v} + \mathbf{a} \cdot \mathbf{p} + \frac{F}{2\tau} < \mathbf{v}, \mathbf{p} > + \eta(p_0 + Sx)\right]\right\} \\
& f(p_0 + \mathbf{v} \cdot \mathbf{p} + \frac{1}{2}\tau\mathbf{v}^2 + Sx, \delta(x)^{-1}(\mathbf{p} + \tau\mathbf{v})).
\end{aligned} \tag{3.49}
$$

The condition (3.5) is fulfilled so V induces a projective representation of $\mathcal{G}_+^\uparrow$.

c,d) These cases correspond to $\tau = 0$. The orbite Z_r^3 and Z_r^5 ($r \neq 0$) can be analysed simultaneously. We have:

$$G_0 = G_{[0,r\mathbf{e_1}]} = \left\{ \left(2\pi n, -\frac{2\pi n S}{r}\mathbf{e_1} + \alpha\mathbf{e_2}\right) |n \in \mathbf{Z}, \alpha \in \mathbb{R}\right\}$$

It is convenient to denote

$$(n, \alpha) \equiv \left(2\pi n, -\frac{2\pi n S}{r}\mathbf{e}_1 + \alpha\mathbf{e}_2\right) \tag{3.50}$$

and to observe that:

$$(n_1, \alpha_1) \cdot (n_2, \alpha_2) = (n_1 + n_2, \alpha_1 + \alpha_2) \tag{3.51}$$

i.e. $G_0 \simeq \mathbf{Z} \times \mathbb{R}$. Next, one computes $m_F|_{G_0 \times G_0}$:

$$m_F((n_1, \alpha_1), (n_2, \alpha_2)) = exp\left\{\frac{ik}{2}(\alpha_1 n_2 - \alpha_2 n_1)\right\} \tag{3.52}$$

where $k \equiv \frac{2\pi F S}{r}$. So, we have two subcases:

(i) $F = 0, \ S \neq 0$ or $F \neq 0, \ S = 0$.

In both situations we have $k = 0$ so $m_F|_{G_0 \times G_0} = 0$. It follws that we have to take into account true representations of the group G_0. They are of the form $D^{(s,t)}$ ($s \in \mathbb{R} \ (\mathrm{mod} \ 1), \ \mathrm{t} \in \mathbb{R}$) and act in $\mathbf{C}$ according to:

$$D^{(s,t)}(n, \alpha) = e^{2\pi i s n} \, e^{it\alpha}. \tag{3.53}$$

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An associated m_F-cocycle can be computed using the formula (2.18). The final expression is rather complicated. Define the auxilliary function:

$$\Phi((x,\mathbf{v}),[p_0,\mathbf{p}]) \equiv \left\{ \frac{iF}{2r^2} <\delta(x)\mathbf{p},\mathbf{v}> \left[\mathbf{v}\cdot\delta(x)\mathbf{p} - 2p_0 + S(c_\mathbf{p} + c_{\delta(x)\mathbf{p}} - x)\right] \right\} \tag{3.54}$$

where $\mathbb{R}^2 \ni \mathbf{p} \mapsto c_\mathbf{p} \in \mathbb{R}$ is a Borel cross section verifying:

$$\delta(c_\mathbf{p})\mathbf{e}_1 = \frac{\mathbf{p}}{r}. \tag{3.55}$$

Then a cocycle associated to $D^{(s,t)}$ can be taken to be:

$$\phi^{(s,t)}((x,\mathbf{v}),[p_0,\mathbf{p}]) = \Phi((x,\mathbf{v}),[p_0,\mathbf{p}])exp\left\{i\left[\frac{k}{r} <\delta(x)\mathbf{p},\mathbf{v}> +2\pi isx\right]\right\}. \tag{3.56}$$

The m_F-system of imprimitivity acts in $L^2(\mathbb{R}\times C_r, \mathrm{d}p_0 \otimes \mathrm{d}\Omega)$ where $C_r \equiv \{\mathbf{p} \in \mathbb{R}^2 | \mathbf{p}^2 = r^2\}$ and $d\Omega$ is the Lebesgue measure on C_r.

We have:

$$(U_{x,\mathbf{v}}\ f)(p_0,\mathbf{p}) = \phi^{(s,t)}((x,\mathbf{v}),(x,\mathbf{v})^{-1}\cdot[p_0,\mathbf{p}])\ f(p_0 + \mathbf{v}\cdot\mathbf{p} + Sx, \delta(x)^{-1}\mathbf{p}) \tag{3.57}$$

$$P_E = \chi_E. \tag{3.58}$$

The corresponding expression for W is

$$(W_{\eta,\mathbf{a}}\ f)(p_0,\mathbf{p}) = e^{i(\eta p_0 + \mathbf{a}\cdot\mathbf{p})}\ f(p_0,\mathbf{p}). \tag{3.59}$$

(ii) $F \neq 0,\ S \neq 0$

In this situation $k \neq 0$ and we have to classify some projective representations of G_0, namely those representations $D_{n,\alpha}$ verifying:

$$D_{n_1,\alpha_1}\ D_{n_2,\alpha_2} = exp\{-ik(\alpha_1 n_2 - \alpha_2 n_1)\}\ D_{n_1+n_2,\alpha_1+\alpha_2} \tag{3.60}$$

It is convenient to define:

$$\mathcal{U}_\alpha \equiv D_{0,\alpha}, \quad \mathcal{W}_n \equiv D_{n,0}. \tag{3.61}$$

Then we have:

Proposition 6 *(i) $\mathcal{U}$ and $\mathcal{W}$ are unitary representations of the Abelian groups $\mathbb{R}$ and $\mathbf{Z}$ respectively, i.e.:*

$$\mathcal{U}_{\alpha_1}\ \mathcal{U}_{\alpha_2} = \mathcal{U}_{\alpha_1+\alpha_2} \tag{3.62}$$

$$\mathcal{W}_{n_1}\ \mathcal{W}_{n_2} = \mathcal{W}_{n_1+n_2}. \tag{3.63}$$

We also have:

$$\mathcal{U}_\alpha\ \mathcal{W}_n\ \mathcal{U}_\alpha^{-1} = e^{-ik\alpha n}\ \mathcal{W}_n. \tag{3.64}$$

(ii) Conversely, if $\mathcal{U}$ and $\mathcal{W}$ are as above and we define

$$D_{n,\alpha} \equiv e^{\frac{i}{2}kn\alpha} \, \mathcal{W}_n \, \mathcal{U}_\alpha \tag{3.65}$$

then D is a projective unitary representation of $\mathbf{Z} \times \mathbb{R}$; more precisely it verifies (3.60).

(iii) Moreover, the commuting ring of D is isomorphic to the commuting ring of the couple $(\mathcal{U}, \mathcal{W})$.

It is almost evident that the couple $(\mathcal{U}, \mathcal{W})$ can be connected to a system of imprimitivity applying SNAG theorem. Indeed, the unitary irreducible representations of $\mathbf{Z}$ are one-dimensional and have the well-known expression:

$$n \mapsto z^n, \quad (z \in \mathbf{T}) \tag{3.66}$$

So $\hat{\mathbf{Z}} \simeq \mathbf{T}$ and SNAG theorem says that the representation $\mathcal{W}$ has the generic form:

$$\mathcal{W}_n = \int_{\mathbf{T}} z^n \, dP(z) \tag{3.67}$$

where P is some projector valued measure on $\mathbf{T}$.

Then (3.64) is shown to be equivalet to:

$$\mathcal{U}_\alpha \, P_E \, \mathcal{U}_\alpha^{-1} = P_{\alpha \cdot E} \tag{3.68}$$

where the action of $\mathbb{R}$ on $\mathbf{T}$ is:

$$\alpha \cdot z \equiv e^{ik\alpha} z. \tag{3.69}$$

So $(\mathcal{U}, \mathcal{P})$ is a system of imprimitivity for the Abelian group $\mathbb{R}$ based on $\mathbf{T}$ and relative to the action (3.69). Moreover $(\mathcal{U}, \mathcal{P})$ is irreducible *iff* the representation D we have started with is irreducible. To classify these irreducible systems of imprimitivity is easy because for $k \neq 0$ they are transitive. One has

$$G_0 = G_{\{1\}} = \left\{ \frac{2\pi n}{k} \Big| n \in \mathbf{Z} \right\}.$$

The unitary irreducible representations of G_0 are one-dimensional; they are indexed by $\lambda \in \mathbb{R}$ (mod 1) and have the expression:

$$\pi^{(\lambda)} \left(\frac{2\pi n}{k} \right) = e^{2\pi i \lambda n}. \tag{3.70}$$

An associated cocycle is:

$$\phi^{(\lambda)}(\alpha, z) = e^{ik\lambda\alpha} \tag{3.71}$$

and the corresponding system of imprimitivity acts in $\mathcal{K} = L^2(\mathbf{T}, d\omega)$ ($d\omega \equiv$ the Lebesgue measure on $\mathbf{T}$) according to:

$$(\mathcal{U}_\alpha \, f)(z) = e^{ik\lambda\alpha} \, f(e^{-ik\alpha} z) \tag{3.72}$$

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$$P_E = \chi_E. \tag{3.73}$$

The expression of $\mathcal{W}$ is obtained from (3.67) and it is:

$$(\mathcal{W}_n \ f)(z) = z^n \ f(z). \tag{3.74}$$

Summing up, it follows that every projective unitary irreducible representation of $\mathbf{Z} \times \mathbb{R}$ verifying (3.60) is unitary equivalent to one of the form $D^{(\lambda)}$ which acts in $\mathcal{K}$ according to

$$\left(D^{(\lambda)}_{n,\alpha} \ f\right)(z) = e^{ik(\lambda - n/2)\alpha} \ z^n \ f(e^{-ik\alpha}z). \tag{3.75}$$

Moreover $D^{(\lambda_1)} \simeq D^{(\lambda_2)}$ iff $\lambda_1 - \lambda_2 \in \mathbf{Z}$.
A corresponding m_F-cocycle is:

$$\left(\phi^{(\lambda)}((x,\mathbf{v}), [p_0, \mathbf{p}]) \ f\right)(z) = \Phi((x,\mathbf{v}), [p_0, \mathbf{p}])e^{2\pi ik\lambda x}z^x \ f\left(e^{-\frac{ik}{r}<\delta(x)\mathbf{p},\mathbf{v}>}z\right) \tag{3.76}$$

where Φ has been defined before (see (3.54)).

So, the looked for m_F-system of imprimitivity (U, P) acts in $L^2(\mathbb{R} \times C_r, \mathrm{dp}_0 \otimes \mathrm{d}\Omega, \mathcal{K}) \simeq L^2(\mathbb{R} \times C_r \times \mathbf{T}, \mathrm{dp}_0 \otimes \mathrm{d}\Omega \otimes \mathrm{d}\omega)$ according to

$$\begin{aligned}(U_{x,\mathbf{v}} \ f)(p_0, \mathbf{p}, z) &= \Phi((x,\mathbf{v}), (x,\mathbf{v})^{-1} \cdot [p_0, \mathbf{p}])e^{-2\pi ik\lambda x} \\ &\quad z^{-x} \ f\left(p_0 + \mathbf{v} \cdot \mathbf{p} + Sx, e^{-\frac{ik}{r}<\delta(x)\mathbf{p},\mathbf{v}>}z\right) \end{aligned} \tag{3.77}$$

$$(P_E \ f)(p_0, \mathbf{p}, z) = \chi_E(p_0, \mathbf{p})f(p_0, \mathbf{p}, z). \tag{3.78}$$

The expression of W is:

$$(W_{\eta,\mathbf{a}} \ f)(p_0, \mathbf{p}, z) = e^{i(\eta p_0 + \mathbf{a}\cdot\mathbf{p})} \ f(p_0, \mathbf{p}, z). \tag{3.79}$$

We still have to analyse the case (d) with $r = 0$. In this case:

$$G_0 = G_{[0,\mathbf{0}]} = \{(0, \mathbf{v})|\mathbf{v} \in \mathbb{R}^2\} \simeq \mathbb{R}^2.$$

We have to classify all the unitary irreducible m_F-representations of the Abelian group $\mathbb{R}^2$ i.e.

$$D_{\mathbf{v}_1} \ D_{\mathbf{v}_2} = e^{-\frac{iF}{2}<\mathbf{v}_1,\mathbf{v}_2>} \ D_{\mathbf{v}_1+\mathbf{v}_2}. \tag{3.80}$$

It is clear that D will be unitary equivalent to a unitary irreducible representation of the canonical commutation relations. So $D \simeq \pi^{CCR} \equiv$ the Schroedinger representation (see e.g. [6]). The corresponding system of imprimitivity acts in the Hilbert space $L^2(\mathbb{R}, \mathrm{dp}_0, \mathcal{K}^{CCR})$ where $\mathcal{K}^{CCR}$ is the representation space of π^{CCR}.

We have explicitly:

$$(U_{x,\mathbf{v}} \ f)(p_0) = \pi^{CCR}_{\mathbf{v}} \ f(p_0 + Sx) \tag{3.81}$$

$$P_E = \chi_E \tag{3.82}$$

which gives:

$$(W_{\eta,\mathbf{a}} \ f)(p_0) = e^{i\eta p_0} \ f(p_0). \tag{3.83}$$

Finally, the trivial orbit $Z^4_{p_0}$ corresponds to m_F-representations of G.

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Remark 1 *The case $\tau \neq 0$ can be simplified somehow if we make the following obsevation. Let V be a $m_{\tau,F,S}$-representation. We define:*

$$V'_{x,\mathbf{v},\eta,\mathbf{a}} \equiv V_{x,\mathbf{v},\eta,\mathbf{a}+\frac{F}{2\tau}A\mathbf{v}}. \tag{3.84}$$

Then V' is a $m_{\tau,0,S}$-representation. So in the case $\tau \neq 0$ one can analyse only the situation $F = 0$. After the list of the irreducible unitary representations V' is obtained, one gets the corresponding V by making $\mathbf{a} \mapsto \mathbf{a} + \frac{F}{2\tau}A\mathbf{v}$.

This remark appears in an infinitesimal form in [7].

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